AN OPTIMAL CONTROL MODEL OF THE LIFE-CYCLE RESEARCH PRODUCTIVITY OF SCIENTISTS

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A continuous time model using optimal control techniques is presented which implies that a scientist's productivity will eventually decline with age. This implication is at variance with *Cole*'s empirical findings[1] but is consistent with *Diamond*'s empirical findings.[2]

In recent work[3] a discrete time model due to Becker was specialized to explain the research productivity of scientists. Here mathematically more sophisticated optimal control techniques[4] are used to more rigorously derive similar implications in a continuous time framework. The model presented here is essentially an adaptation of a simplified version of a model first presented by *Ben-Porath.*[5]

At any age i, the model has the scientist maximizing U, the present value of his disposable income:

$$U = \int_{i}^{n} e^{-rt}[W K(t) - W S_t K(t)]dt \tag{1}$$

where K is his stock of prestige capital, W is the rate of return to prestige capital, i is the scientist's current age, n is the scientist's age at retirement, and S_t is the proportion of time in period t that is devoted to the production of prestige capital. S_t is constrained to be in the closed interval [0,1]. In order to satisfy the Weierstrass condition for a maximum we assume that the integrand is a concave function of S_t.

Let the production function for prestige capital be:

$$Q_t = \beta (S_t K_t)^{\alpha} \tag{2}$$

where Q_t is the flow of prestige capital. For some purposes the inclusion of non-time inputs is important, but we ignore them nere.

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The rate of change of the capital stock is given by:

$$\dot{K}_t = Q_t - \delta K_t \tag{3}$$

where a dot above the K indicates a derivative with respect to time and where δ is the rate by which the stock of prestige capital deteriorates.

The Hamiltonian for this problem is:

$$H = e^{-rt}(1 - S)\, W K + \lambda(Q - \delta K) \tag{4}$$

where the costate variable λ is the discounted shadow price of investment in prestige capital. The first of the relevant necessary conditions for a maximum is:

$$\dot{\lambda} = \frac{-\partial H}{\partial K} = -e^{-rt}(1 - S)\, W - \lambda\, (\alpha \frac{Q}{K} - \delta) \tag{5}$$

In terms of current prices $\mu = \lambda\, e^{-rt}$. So the condition can be re-written as:

$$\dot{\mu} = -(1 - S)\, W - \mu\, (\alpha \frac{Q}{K} - \delta - r) \tag{6}$$

The second of the relevant necessary conditions for a maximum is:

$$0 = \frac{\partial H}{\partial S} = -e^{-rt}\, W K + \mu\, e^{-rt}\, \alpha \frac{Q}{S} \tag{7}$$

The transversality condition is that:

$$\lambda\, (n) = 0 \tag{8}$$

Substituting Eq. (2) into Eq. (7) we can eventually obtain:

$$Q = (\beta)^{\frac{1}{1-\alpha}} \left(\frac{\mu a}{W}\right)^{\frac{\alpha}{1-\alpha}} \tag{9}$$

Substituting Eq. (7) into Eq. (6) we obtain the simple differential equation:

$$\dot{\mu} = -W + \mu\, (\delta + r) \tag{10}$$

After application of the transversality condition, the solution to (10) is:

$$\mu = \frac{W}{\delta + r} \left[1 - e^{-(\delta + r)(n - i)}\right] \tag{11}$$

In order to obtain a solution for Q in terms of the parameters of the model, Eq. (11) can be substituted into Eq. (9) in order to obtain:

$$Q = (\beta)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{\delta + r}\right)^{\frac{\alpha}{1-\alpha}} \left[1 - e^{-(\delta + r)(n - i)}\right]^{\frac{\alpha}{1-\alpha}} \tag{12}$$

When a scientist dies $n = i$, which from Eq. (12) implies that at the end of life $Q = 0$. For finite n, Q declines continuously as the scientist ages (i.e., as i increases).

The life-cycle profile for prestige capital can be inferred from Eq. (3). Initially Q_t will be relatively high and K_t will be relatively low, so $\dot{K}_t$ will be positive. As Q_t declines and K_t rises, eventually $\dot{K}_t$ will be negative. The life-cycle profile is thus concave.

A continuous time model, like the earlier discrete time model, implies that scientific productivity eventually declines as a scientist grows older. This implication is at variance with evidence reported by *Cole*[1] but is consistent with evidence reported by *Diamond.*[2]

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I have benefitted from conversations with Richard *Cantor* and Donald *Parsons.* Unpublished notes by Jong C. *Rhee* also proved useful.

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