Technical Appendices

Appendix A. Derivation of Condition Equity Premium (Equation (11)).

Recall the return on equity is:

$$\frac{P_{t+1} + D_{t+1}}{P_t} = \left(\frac{D_t}{P_t}\right)\left(\frac{D_{t+1}}{D_t}\right)\left(1 + \frac{P_{t+1}}{D_{t+1}}\right).$$

Taking logarithms and using the log approximation employed in the text

$$\text{(A1)} \quad \log\left(\frac{P_{t+1} + D_{t+1}}{P_t}\right) = -\log\left(\frac{P_t}{D_t}\right) + \log\left(\frac{D_{t+1}}{D_t}\right) + \rho\log\left(\frac{P_{t+1}}{D_{t+1}}\right) + k.$$

Define $(1 + r_t^f)$ as the (gross) risk free interest rate. Note that $(1 + r_t^f) = \frac{1}{E_t(M_{t+1})}$. Consider the logarithm of the conditional equity premium

$$\text{(A2)} \quad \log(1 + EP) = \log\left(E_t\left(\left(\frac{P_{t+1} + D_{t+1}}{P_t}\right)/(1 + r_t^f)\right)\right)$$

$$= \log\left(E_t\left(\frac{P_{t+1} + D_{t+1}}{P_t}\right)\right) + \log(E_t M_{t+1})$$

Assuming log normality:

$$\text{(A3)} \quad \log(1 + EP) = -\log\left(\frac{P_t}{D_t}\right) + E_t\log\left(\frac{D_{t+1}}{D_t}\right) + \rho E_t\log\left(\frac{P_{t+1}}{D_{t+1}}\right) + k$$

$$+\frac{1}{2}\text{Var}_t\left(\log\left(\frac{D_{t+1}}{D_t}\right) + \rho\log\left(\frac{P_{t+1}}{D_{t+1}}\right)\right) + E_t\log(M_{t+1}) + \frac{1}{2}\text{Var}_t(\log(M_{t+1})).$$

Using equation (10) in the text to substitute for $\log\left(\frac{P_t}{D_t}\right)$ in equation (A3), yields

$$\text{(A4)} \quad \log(1 + EP) = -\text{Cov}_t\left(\log(M_{t+1}),\ \log\left(\frac{D_{t+1}}{D_t}\right) + \rho\log\left(\frac{P_{t+1}}{D_{t+1}}\right)\right).$$

Appendix B: Calculate implied VAR ratio from our model

Variance ratio implied by our model as k tends to infinity is given by:

(B1) $$\lim_{k\to\infty} \frac{\dfrac{\operatorname{var}\left(d_{t+k}-d_t\right)}{k}}{\operatorname{var}\left(d_{t+1}-d_t\right)} = \frac{\operatorname{var}\left(v_t^{df}\right)}{\operatorname{var}\left(d_{t+1}-d_t\right)} = \frac{.0875}{3.1512} = .028$$

Recall that $d_t = d_t^f + d_t^a$. The variance of Δd_t is given by:

(B2) $$\operatorname{var}\left(\Delta d_t\right) = \operatorname{var}\left(v_t^{df}\right) + 2\operatorname{cov}\left(v_t^{df}\ d_t^a\right) + 2\operatorname{var}\left(d_t^a\right) - 2\operatorname{cov}\left(d_t^a d_{t-1}^a\right)$$

Note that $\operatorname{cov}\left(v_t^f\ d_{t-1}^a\right) = 0$, and $\operatorname{var}\left(d_t^a\right) = \operatorname{var}\left(d_{t-1}^a\right)$ and $\operatorname{cov}\left(d_t^a\ d_{t-1}^a\right) = \operatorname{cov}\left(d_{t-1}^a\ d_{t-2}^a\right)$ as $\mathbf{d_t^a}$ is stationary. Recall also that

(B3) $$d_t^a = \theta_1 d_{t-1}^a + \theta_2 d_{t-2}^a + V_t^{da} \quad .$$

Thus,

(B4) $$\operatorname{var}\left(d_t^a\right) = \left(\theta_1^2\right)\operatorname{var}\left(d_t^a\right) + \left(\theta_2^2\right)\operatorname{var}\left(d_t^a\right) + 2\theta_1\theta_2\ \operatorname{cov}\left(d_t^a d_{t-1}^a\right) + \operatorname{var}\left(v_t^{da}\right)$$

(B5) $$\operatorname{cov}(d_t^a, d_{t-1}^a) = \frac{\theta_1 \operatorname{var}(d_t^a)}{(1-\theta_2)}.$$

Using (B4) and (B5) to solve for $\operatorname{var}(d_t^a)$, yields

(B6) $$\operatorname{var}(d_t^a) = \frac{\operatorname{var}(v_t^{da})}{(1-\theta_2)(1-\theta_1^2-\theta_2^2)-2\theta_1^2\theta_2}.$$

Substituting in (B2) yields

(B7) $$\operatorname{var}(\Delta d_t) = \operatorname{var}(v_t^{df}) + 2\operatorname{cov}(v_t^{df}, v_t^{da}) + \frac{2(1-\theta_1-\theta_2)\operatorname{var}(v_t^{da})}{(1-\theta_2)[(1-\theta_2)(1-\theta_1^2-\theta_2^2)-2\theta_1^2\theta_2]}.$$

Appendix C. Implications for future expected real returns.

The gross (one period) return on equity is defined as $R^{e}_{t+1} = \left(\frac{P_{t+1} + D_{t+1}}{P_t} \right)$. Taking a log approximation yields,

$$\log(R^{e}_{t+1}) = \log\left(\frac{D_{t+1}}{D_t} \right) - \log\left(\frac{P_t}{D_t} \right) + \rho \log\left(\frac{P_{t+1}}{D_{t+1}} \right) + k,$$

where $\rho = \exp(\log(P/D)^{avg})/(1+\exp(\log(P/D)^{avg}))$ and

$k = \log(1+\exp(\log(P/D)^{avg})) - \rho \log(P/D)^{avg}$ with $\log(P/D)^{avg}$ being the average of the log price-dividend ratio over the sample.

Using equation (10) to substitute in for $\log\left(\frac{P_t}{D_t} \right)$, we obtain:

$$\text{(C1)} \quad \log(R^{e}_{t+1}) \approx - E_t \log(M_{t+1}) + \log\left(\frac{D_{t+1}}{D_t} \right) - E_t \log\left(\frac{D_{t+1}}{D_t} \right)$$
$$+ \rho \left[\log\left(\frac{P_{t+1}}{D_{t+1}} \right) - E_t\left(\log\left(\frac{P_{t+1}}{D_{t+1}} \right) \right) \right]$$
$$- \frac{1}{2} VAR_t(\log(M_{t+1})) - \frac{1}{2} VAR_t\left(\log\left(\frac{D_{t+1}}{D_t} \right) + \rho \log\left(\frac{P_{t+1}}{D_{t+1}} \right) \right)$$
$$- COV_t\left(\log(M_{t+1}),\ \log\left(\frac{D_{t+1}}{D_t} \right) + \rho \log\left(\frac{P_{t+1}}{D_{t+1}} \right) \right)$$

Note that expected real returns (conditioned on time t information) will depend only on the conditional expectation of the discount factor and the variance/covariance terms in equation

(C1).

We can also write the log gross real returns of stocks in terms of our state space model:

(C2) $$\log(R^e_{t+1}) = H_3 S_{t+1} - [H_1(I-\rho F)^{-1}F - H_{ep}(I-\rho F)^{-1}]S_t + \rho[H_1(I-\rho F)^{-1}F - H_{ep}(I-\rho F)^{-1}]S_{t+1},$$

where

$H_1 = (1,1,0,1,1,0,0,0,0_{1x5(m-1)})$, $H_{ep} = (0,0,0,0,0,0,1,0,0_{1x5(m-1)})$, and

$H_3 = (1,0,0,1,0,0,0,0,0_{1x5(m-1)})$ with m equal to number of lags for the partial adjustment factors (d_t^a, m_t^a, π_t^a) and the equity and term premium factors (ep_t, tp_t).

The average multi-period return is given by $R_{n,t} = (\prod_{i=1}^{n} R_{t+i})^{1/n}$. The expected multi-period (log) returns is:

$$E_t\log(R_{n,t}) = \frac{1}{n}\sum_{i=1}^{n} E_t\log(R_{t+i}).$$

Using the state space model, the multi-period real return for equity is given by:

$$E_t\log(R^e_{n,t}) = \frac{1}{n}\sum_{i=1}^{n} E_t[H_3 S_{t+i} - [H_1(I-\rho F)^{-1}F - H_{ep}(I-\rho F)^{-1}]S_{t+i-1} + \rho[H_1(I-\rho F)^{-1}F - H_{ep}(I-\rho F)^{-1}]S_{t+i}]$$

$$= \frac{1}{n}\sum_{i=1}^{n} [H_3 F^i - [H_1(I-\rho F)^{-1}F - H_{ep}(I-\rho F)^{-1}]F^{i-1} + \rho[H_1(I-\rho F)^{-1}F - H_{ep}(I-\rho F)]F^i]S_{t|t},$$

where $S_{t|t}$ is the conditional expectation of S_t given information at time t. After more algebra, expected n-period real return is:

(C3) $$E_t\log(R^e_{n,t}) = \frac{1}{n}(H_3 - H_1)(I-F^n)(I-F)^{-1}F S_{t|t} + \frac{1}{n}H_{ep}(I-F^n)(I-F)^{-1}S_{t|t},$$

The term $\frac{1}{n}(H_3 - H_1)(I-F^n)(I-F)^{-1}F S_{t|t}$ in equation (C3) is just the expected (minus) future n-period discount factor (see equation (C1) above). The term $\frac{1}{n}H_{ep}(I-F^n)(I-F)^{-1}S_{t|t}$ in equation

(C3) is the effect of the equity premium factor, ep_t, in the log(P/D) equation.

Similarly, we can calculate the multi-period real returns for long bonds (held to maturity):

$$\text{(C4)} \quad E_t \log(R^{lb}_{n,t}) \;=\; \frac{1}{n}(H_3 - H_1)(I - F^{\,n})(I - F)^{-1} F\,S_t \;+\; H_{tp} S_t\,,$$

where $H_{z2} \;=\; (0,0,0,0,0,0,0,1,0_{1x5(m-1)})$. $H_{tp} S_t$ is the contribution of the term premium factor in the long-term interest rate equation. Finally, the expected real return to holding a sequence of short bonds over n-periods is:

$$\text{(C5)} \quad E_t \log(R^{sb}_{n,t}) \;=\; \frac{1}{n}(H_3 - H_1)(I - F^{\,n})(I - F)^{-1} F\,S_t\,.$$

Thus, all expected multi-period real asset returns share the term $\frac{1}{n}(H_3 - H_1)(I - F^{\,n})(I - F)^{-1} F\,S_t$ which is the expected future n-period discount factor.

Because we demeaned the data before estimating the state-space model, to calculate the expected real returns displayed in Figures 10 and 11 we added back the sample means of the variables, thus:

$$\text{(C6)} \quad E_t \log(R^{e}_{n,t}) \;=\; \frac{1}{n}(H_3 - H_1)(I - F^{\,n})(I - F)^{-1} F\,S_t \;+\; \frac{1}{n} H_{ep} (I - F^{\,n})(I - F)^{-1} S_t$$
$$\;+\; \log\left(D_{t+1}/D_t\right)^{avg} \;-\; (1 - \rho) \log\left(P_t/D_t\right)^{avg} \;+\; k$$

$$\text{(C7)} \quad E_t \log(R^{lb}_{n,t}) \;=\; \frac{1}{n}(H_3 - H_1)(I - F^{\,n})(I - F)^{-1} F\,S_t \;+\; H_{tp} S_t \;+\; \left(i_t^{\,n}\right)^{avg} \;-\; \pi_t^{\,avg},$$

$$\text{(C8)} \quad E_t \log(R^{sb}_{n,t}) \;=\; \frac{1}{n}(H_3 - H_1)(I - F^{\,n})(I - F)^{-1} F\,S_t \;+\; \left(i_t^{\,1}\right)^{avg} \;-\; \pi_t^{\,avg},$$

where $\log(D_{t+1}/D_t)^{avg}$, $\log(P_t/D_t)^{avg}$, $(i_t^{\,n})^{avg}$, $(i_t^{\,1})^{avg}$, and $\pi_t^{\,avg}$ are the sample averages of dividend growth, log price-dividend ratio, long-term interest rates, short-term interest rates, and inflation, respectively.

Appendix D: Proof of Identification

Consider state space model given in equations (23) and (31). Rewriting these yields:

(D1) $Y_t = H\,S_t + W_t$

(D2) $S_t = F\,S_{t-1} + G\,v_t$

where $v_t = (v_t^{df}, v_t^{mf}, v_t^{\pi f}, v_t^{da}, v_t^{ma}, v_t^{\pi a})'$, with $E[v_t\, v_t'] = Q$, $E[W_t\, W_t'] = R$, and

$G = \begin{bmatrix} I_{6x6} \\ 0_{3(m-1)\,x6} \end{bmatrix}$. Q and R are full rank.

In our estimated model we set m=2, so that the dimension of the state vector is 9x1.

Proof of identification follows that outlined in Burmeister, Hamilton, and Wall (1985) and Wall (1987). First, we show that the state space model is minimal. This requires the model satisfy controllability and observability conditions. Define C = [G FG . . . F^KG . . . . .]. Controllability requires the matrix C be of rank equal to number of state variables (here 9) (see Aoki (1987)).

Consider the following sub matrix of C: [G FG]. For the case where m=2,

$$[G \quad FG] = \begin{bmatrix} I_{3x3} & 0_{3x3} & I_{3x3} & 0_{3x3} \\ 0_{3x3} & I_{3x3} & 0_{3x3} & \theta_1 \\ 0_{3x3} & 0_{3x3} & 0_{3x3} & I_{3x3} \end{bmatrix}$$

By inspection, we see that this matrix has rank = 9. Thus, the state space model satisfies the controllability condition.

Define the observability matrix O, O = [H' F'H' $(F')^2$H' . . . $(F')^n$H' . . .]

Observability requires that matrix O to be of rank equal to number of state variables (which in the basic model is equal to nine). Recall, for m = 2

$$\text{(D3)} \qquad H = \begin{bmatrix} H_1(I-\rho F)^{-1}F \\ H_2\left(\frac{1}{n}\right)(I-F^n)(I-F)^{-1}F \\ H_2F \\ H_3 \\ H_4 \end{bmatrix} \quad \begin{array}{l} H_1 = (1,1,0,1,1,0,0,0,0) \\ H_2 = (0,-1,1,0,-1,1,0,0,0) \\ H_3 = (1,0,0,1,0,0,0,0,0) \\ H_4 = (0,0,1,0,0,1,0,0,0) \end{array} .$$

After some tedious algebra one can show:

$$(D4) \qquad (I-\rho F)^{-1}F = \left[\begin{array}{c:c:c} \frac{1}{1-\rho}I_{3x3} & 0_{3x3} & 0_{3x3} \\ \hdashline 0_{3x3} & (I-\rho\theta_1-\rho^2\theta_2)^{-1}(\theta_1+\rho\theta_2) & (I-\rho\theta_1-\rho^2\theta_2)^{-1}\theta_2 \\ \hdashline 0_{3x3} & (I-\rho\theta_1-\rho^2\theta_2)^{-1} & (I-\rho\theta_1-\rho^2\theta_2)^{-1}\rho\theta_2 \end{array}\right]$$

and

$$(D5) \qquad \frac{1}{n}(I - F^n)(I - F)^{-1}F$$

$$= \left[\begin{array}{c|c|c} I_{3x3} & 0_{3x3} & 0_{3x3} \\ \hline 0_{3x3} & \frac{1}{n}\left[\left(I - C_{11}^{(n)} - C_{12}^{(n)}\right)\theta_1 + (I - C_{11}^{(n)})\theta_2 - C_{12}^{(n)}(I - \theta_1)\right](I - \theta_1 - \theta_2)^{-1} & \frac{1}{n}(I - C_{11}^{(n)} - C_{12}^{(n)})(I - \theta_1 - \theta_2)^{-1}\theta_2 \\ \hline 0_{3x3} & \frac{1}{n}\left[\left(I - C_{21}^{(n)} - C_{22}^{(n)}\right)\theta_1 - C_{21}^{(n)}\theta_2 + (I - C_{22}^{(n)})(I - \theta_1)\right](I - \theta_1 - \theta_2)^{-1} & \frac{1}{n}(I - C_{21}^{(n)} - C_{22}^{(n)})(I - \theta_1 - \theta_2)^{-1}\theta_2 \end{array}\right]$$

where

$$\left[\begin{array}{c|c} C_{11}^{(n)} & C_{12}^{(n)} \\ \hline C_{21}^{(n)} & C_{22}^{(n)} \end{array}\right] = \left[\begin{array}{c|c} \theta_1 & \theta_2 \\ \hline I_{3x3} & 0_{3x3} \end{array}\right]^n.$$ $C_{ij}^{(n)}$ is a 3x3 diagonal matrix denoted by:

$$C_{ij}^{(n)} = \begin{bmatrix} C_{ij}^{(n)d} & 0 & 0 \\ 0 & C_{ij}^{(n)m} & 0 \\ 0 & 0 & C_{ij}^{(n)d} \end{bmatrix}.$$

After more algebra, we can write

$$
\begin{bmatrix} H' & F'H' \end{bmatrix} = \begin{bmatrix}
\frac{1}{1-\rho} & 0 & 0 & 1 & 0 & \frac{1}{1-\rho} & 0 & 0 & 1 & 0 \\
\frac{1}{1-\rho} & -1 & -1 & 0 & 0 & \frac{1}{1-\rho} & -1 & -1 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
H_{14} & 0 & 0 & 1 & 0 & \theta_1^{d} H_{16} + H_{17} & 0 & 0 & \theta_1^{d} & 0 \\
H_{15} & H_{25} & H_{35} & 0 & 0 & \theta_1^{m} H_{15} + H_{18} & \theta_1^{m} H_{25} + H_{28} & \theta_1^{m} H_{35} + H_{38} & 0 & 0 \\
0 & H_{26} & H_{36} & 0 & 1 & 0 & \theta_1^{\pi} H_{26} + H_{29} & \theta_1^{\pi} H_{36} + H_{39} & 0 & \theta_1^{\pi} \\
H_{17} & 0 & 0 & 0 & 0 & \theta_2^{d} H_{14} & 0 & 0 & \theta_2^{d} & 0 \\
H_{18} & H_{28} & H_{38} & 0 & 0 & \theta_2^{m} H_{15} & \theta_2^{m} H_{25} & \theta_2^{m} H_{35} & 0 & 0 \\
0 & H_{29} & H_{39} & 0 & 0 & 0 & \theta_2^{\pi} H_{26} & \theta_2^{\pi} H_{36} & 0 & \theta_2^{\pi}
\end{bmatrix}
$$

where

$$H_{14} = \left(1-\rho\theta_1^{d} - \rho^2\theta_2^{d}\right)^{-1}\left(\theta_1^{d} + \rho\theta_2^{d}\right)$$

$$H_{15} = \left(1-\rho\theta_1^{m} - \rho^2\theta_2^{m}\right)^{-1}\left(\theta_1^{m} + \rho\theta_2^{m}\right)$$

$$H_{17} = \left(1-\rho\theta_1^{d} - \rho^2\theta_2^{d}\right)^{-1}\theta_2^{d}$$

$$H_{18} = \left(1-\rho\theta_1^{m} - \rho^2\theta_2^{m}\right)^{-1}\theta_2^{m}$$

$$H_{25} = -\left(\frac{1}{n}\right)[(1-C_{11}^{(n)m} - C_{12}^{(n)m})(\theta_1^{m} + (1-C_{11}^{(n)m})\theta_2^{m} - C_{12}^{(n)m}(1-\theta_1^{m})](1-\theta_1^{m} - \theta_2^{m})^{-1}$$

$$H_{26} = \left(\frac{1}{n}\right)[(1-C_{11}^{(n)\pi} - C_{12}^{(n)\pi})(\theta_1^{\pi} + (1-C_{11}^{(n)\pi})\theta_2^{\pi} - C_{12}^{(n)\pi}(1-\theta_1^{\pi})](1-\theta_1^{\pi} - \theta_2^{\pi})^{-1}$$

$$H_{28} = -\left(\frac{1}{n}\right)\left(1-C_{11}^{(n)m} - C_{12}^{(n)m}\right)\left(1-\theta_1^{m} - \theta_2^{m}\right)^{-1}\theta_2^{m}$$

$$H_{29} = \left(\frac{1}{n}\right)\left(1-C_{11}^{(n)\pi} - C_{12}^{(n)\pi}\right)\left(1-\theta_1^{\pi} - \theta_2^{\pi}\right)^{-1}\theta_2^{\pi}$$

$$H_{35} = -\theta_1^{m}$$

$$H_{36} = \theta_1^{\pi}$$

$$H_{38} = -\theta_2^{m}$$

$$H_{39} = \theta_2^{\pi}$$

The matrix [H'   F' H'] will have rank = 9 if $\theta_1^i + \theta_2^i < 1$ i = d, m, π

and $\theta_j^i \neq 0$ i = d, m, π j = 1, 2.

We now show that this minimal state space model is identified. Consider two separate state space vectors $S_{(2)}$ and $S_{(1)}$, where $S_{(2)} = T\, S_{(1)}$. T is a square, nonsingular matrix. These two state spaces are observationally equivalent if:

$T\, F^{(1)} = F^{(2)}T$ $\qquad H^{(1)} = H^{(2)}T$

$T\, G^{(1)} = G^{(2)}$ $\qquad Q^{(1)} = Q^{(2)}$ $\qquad R^{(1)} = R^{(2)}$

If T = I, then $S_{(2)} = S_{(1)}$ and the state space model is globally identified.

It will be convenient to partition the T matrix in the following way:

$$T = \left[\begin{array}{c|c} T_1 & T_2 \\ \hline T_3 & T_4 \end{array}\right]$$

where T_1 is a 3x3 matrix, T_2 is a 3x6 matrix, T_3 is a 6x3 matrix, and T_4 is a 6x6 matrix.

Similarly, partition the F matrix:

$$F = \left[\begin{array}{c|c} I_{3x3} & 0_{3x3} \\ \hline 0_{6x3} & C \end{array}\right] \text{ where } C = \left[\begin{array}{c|c} \theta_1 & \theta_2 \\ \hline I_{3x3} & 0_{3x3} \end{array}\right].$$

Consider the condition $T\, F^{(1)} = F^{(2)}\, T$. This implies:

$$\left[\begin{array}{c|c} T_1 & T_2C^{(1)} \\ \hline T_3 & T_4C^{(1)} \end{array}\right] = \left[\begin{array}{c|c} T_1 & T_2 \\ \hline C^{(2)}T_3 & C^{(2)}T_4 \end{array}\right].$$

Thus, $T_2C^{(1)} = T_2$, $T_3 = C^{(2)}T_3$, and $T_4C^{(1)} = C^{(2)}T_4$.

Partitioning T_2,

$$T_2 = \left[T_1^2 \;\; T_2^2\right],$$

where $T_j^{\,i}$, j = 1, 2 and i = 2 are 3x3 matrices. The condition $T_2C^{(1)} = T_2$ yields

$$\left[T_1^2\ \theta_1^{(1)} + T_2^2 \quad T_1^2\ \theta_2^{(1)}\right] = \left[T_1^2 \quad T_2^2\right]$$

This implies $T_1^2\theta_2^{(1)} = T_2^2$ which in turn implies $T_1^2(I - \theta_1^{(1)} - \theta_1^{\,2}) = 0$. As long as

$\left(\theta_1^{(1)} + \theta_2^{(1)} \neq I\right)$, then $T_1^2 = 0$ and $T_2^2 = 0$. Thus, $T_2 = 0$.

Turning to T_3, partition this matrix into:

$$T_3 = \begin{bmatrix} T_1^3 \\ T_2^3 \end{bmatrix} \quad T_j^3 \ \textit{is a} 3x3 \ \textit{matrix.}$$

Recall that $T_3 = C^{(2)}T_3$. This implies:

$$\begin{bmatrix} T_1^3 \\ T_2^3 \end{bmatrix} = \begin{bmatrix} \theta_1^{(2)}T_1^3 + \theta_2^{(2)}T_2^3 \\ T_1^3 \end{bmatrix}.$$

Thus, $\left(I - \theta_1^{(2)} - \theta_2^{(2)}\right)T_1^3 = 0$. Again, as long as $\theta_1^{(2)} + \theta_2^{(1)} \neq I$, then $T_1^3 = T_2^3 = 0$.

As a result, $T_3 = 0$.

Now, consider $H^{(1)} = H^{(2)}T$. Partition H matrix:

$$H = \begin{pmatrix} (1-\rho)^{-1}H_{11} & H_{12}(I-\rho C)^{-1}C \\ H_{21} & H_{22}\left(\frac{1}{n}\right)(I - C^n)(I-C)^{-1}C \\ H_{21} & H_{22}C \\ H_{31} & H_{32} \\ H_{41} & H_{42} \end{pmatrix}$$

where

$H_j = [H_{j1\,(1x3)} \qquad H_{j2\,(1x6)}] \quad j = 1, \ldots, 4$

for H_j defined in equations (24), (27), (29), and (30) in the text.

Thus, $H^{(2)}T$ is given by:

$$H^{(2)}T = \begin{bmatrix} (1-\rho)^{-1}H_{11}^{(2)}T_1 + H_{12}^{(2)}\left(I-\rho C^{(2)}\right)^{-1}C^{(2)}T_3 & (1-\rho)^{-1}H_{11}^{(2)}T_2 + H_{12}^{(2)}\left(I-\rho C^{(2)}\right)^{-1}C^{(2)}T_4 \\ H_{21}^{(2)}T_1 + H_{22}^{(2)}\left(\frac{1}{n}\right)\left(I-C^{(2)n}\right)\left(I-C^{(2)}\right)^{-1}C^{(2)}T_3 & H_{21}^{(2)}T_2 + H_{22}^{(2)}\left(\frac{1}{n}\right)\left(I-C^{(2)n}\right)\left(I-C^{(2)}\right)^{-1}C^{(2)}T_4 \\ H_{21}^{(2)}T_1 + H_{22}^{(2)}C^{(2)}T_3 & H_{21}^{(2)}T_2 + H_{22}^{(2)}C^{(2)}T_4 \\ H_{31}^{(2)}T_1 + H_{32}^{(2)}C^{(2)}T_3 & H_{31}^{(2)}T_2 + H_{32}^{(2)}C^{(2)}T_4 \\ H_{41}^{(2)}T_1 + H_{42}^{(2)}C^{(2)}T_3 & H_{41}^{(2)}T_1 + H_{42}^{(2)}C^{(2)}T_4 \end{bmatrix}.$$

Write the elements of the T_1 matrix as

$$T_1 = \begin{bmatrix} T_{11}^1 & \cdots & T_{13}^1 \\ \vdots & \ddots & \vdots \\ T_{31}^1 & \cdots & T_{33}^1 \end{bmatrix}.$$

Because $T_3 = 0$, $H^{(2)}T = H^{(1)}$ implies:

$$(1-\rho)^{-1}H_{11}^{(2)}T_1 = (1-\rho)^{-1}H_{11}^{(1)} \Rightarrow \left(T_{11}^1 + T_{21}^1,\ T_{12}^1 + T_{22}^1,\ T_{13}^1 + T_{23}^1\right) = (1,1,0),$$
$$H_{21}^{(2)}T_1 = H_{21}^{(1)} \Rightarrow \left(-T_{21}^1 + T_{31}^1,\ -T_{22}^1 + T_{32}^1,\ -T_{23}^1 + T_{33}^1\right) = (0,-1,1),$$
$$H_{31}^{(2)}T_1 = H_{31}^{(1)} \Rightarrow \left(T_{11}^1,\ T_{12}^1,\ T_{13}^1\right) = (1,0,0),$$
$$H_{41}^{(2)}T_1 = H_{41}^{(1)} \Rightarrow \left(T_{31}^1,\ T_{32}^1,\ T_{33}^1\right) = (0,0,1).$$

Thus, $T_{11}^1 = T_{22}^1 = T_{33}^1 = 1$ and $T_{ij}^1 = 0 \quad i \neq j$. So $T_1 = I_{3x3}$

The condition $H^{(2)}T = H^{(1)}$ also implies:

$$H_{12}^{(2)}(I-\rho C^{(2)})^{-1}C^{(2)}T_4 = H_{12}^{(1)}(I-\rho C^{(1)})C^{(1)},$$
$$H_{22}^{(2)}\left(\frac{1}{n}\right)\left(I-C^{(2)n}\right)\left(I-C^{(2)}\right)^{-1}C^{(2)}T_4 = H_{22}^{(1)}\left(\frac{1}{n}\right)\left(I-C^{(1)n}\right)\left(I-C^{(1)}\right)^{-1}C^{(1)},$$
$$H_{22}^{(2)}C^{(2)}T_4 = H_{22}^{(1)}C^{(1)},$$
$$H_{32}^{(2)}T_4 = H_{32}^{(1)},$$
$$H_{42}^{(2)}T_4 = H_{42}^{(1)}.$$

Denote $T_4 = \begin{pmatrix} T_{11}^4 & \cdots & T_{16}^4 \\ \vdots & & \vdots \\ T_{61}^4 & \cdots & T_{66}^4 \end{pmatrix}$.

Now,

$$H_{32}^{(2)}\ T_4 = \left(T_{11}^4 \cdots T_{16}^4\right) = (1,0,\cdots,0) = H_{32}^{(1)}\ \cdot$$

This implies $T_{11}^4 = 1,\ T_{1j}^4 = 0 \quad j \neq 1.$

Similarly, $H_{42}^{(2)}\ T_4 = H_{42}^{(1)}$ implies $T_{33}^4 = 1,\ T_{3j}^4 = 0 \quad j \neq 3.$

Also, recall that

$$H_{22}^{(2)}C^{(2)}T_4 = H_{22}^{(2)}T_4C^{(1)} = H_{22}^{(1)}C^{(1)},$$

which implies

$$\left[H_{22}^{(2)}T_4 - H_{22}^{(1)}\right]C^{(1)} = 0.$$

As long as $\theta_1^{(1)},\ \theta_2^{(1)} \neq 0$, then $T_{2j}^4 = 0,\ j \neq 2$ with $T_{22}^4 = 1.$

Finally, recall $T_4C^{(1)} = C^{(2)}T_4$.

$$T_4C^{(1)} = \begin{pmatrix} T^4_{11}\theta_1^{d(1)} + T^4_{14} & T^4_{12}\theta_1^{m(1)} + T^4_{15} & T^4_{13}\theta_1^{\pi(1)} + T^4_{16} & T^4_{11}\theta_2^{d(1)} & T^4_{12}\theta_2^{m(1)} & T^4_{13}\theta_2^{\pi(1)} \\ . & . & . & . & . & T^4_{23}\theta_2^{\pi(1)} \\ . & . & . & . & . & T^4_{33}\theta_2^{\pi(1)} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ T^4_{61}\theta_1^{d(1)} + T^4_{64} & T^4_{62}\theta_1^{m(1)} + T^4_{65} & T^4_{63}\theta_1^{\pi(1)} + T^4_{66} & T^4_{61}\theta_2^{d(1)} & T^4_{62}\theta_2^{m(1)} & T^4_{63}\theta_2^{\pi(1)} \end{pmatrix}$$

$$C^{(2)}T_4 = \begin{pmatrix} \theta_1^{d(2)}T^4_{11} + T^4_{41}\theta_2^{d(2)} & . \; . \; . \; . & \theta_1^{d(2)}T^4_{16} + T^4_{46}\theta_2^{d(2)} \\ \theta_1^{m(2)}T^4_{21} + T^4_{51}\theta_2^{m(2)} & . \; . \; . \; . & \theta_1^{m(2)}T^4_{26} + T^4_{56}\theta_2^{m(2)} \\ \theta_1^{\pi(2)}T^4_{31} + T^4_{61}\theta_2^{\pi(2)} & . \; . \; . \; . & \theta_1^{\pi(2)}T^4_{36} + T^4_{66}\theta_2^{\pi(2)} \\ T^4_{11} & . \; . \; . \; . & T^4_{16} \\ T^4_{21} & . \; . \; . \; . & T^4_{26} \\ T^4_{31} & . \; . \; . \; . & T^4_{36} \end{pmatrix}.$$

Using the condition $T_4C^{(1)} = C^{(2)}T_4$, we can show $T^4_{4j} = 0$ for $j \neq 4$, $T^4_{44} = 1$,

$T^4_{55} = 0$ for $j \neq 5$, $T_{55} = 1$, $T^4_{6j} = 0$ for $j \neq 6$, and $T^4_{66} = 1$

Summarizing,

$T_1 = I_{3x3}$, $T_4 = I_{6x6}$, $T_2 = 0_{3x6}$, $T_3 = 0_{6x3}$, which implies that T = I. Thus, the model is identified.